An Improved Digital Algorithm for Fast Amplitude Approximations of Quadrature Pairs

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The authors have discovered a computationally fast algorithm for approximating the amplitude $A = \sqrt{I^2 + Q^2}$ of a quadrature pair (I,Q); specifically, the piecewise linear formula

$$\hat{A} = \begin{cases} X + \frac{1}{8}Y; & X \geqslant 3Y \\ \frac{7}{8}X + \frac{1}{2}Y; & X \leqslant 3Y \end{cases}$$

where $X \equiv max(|I|, |Q|)$, $Y \equiv min(|I|, |Q|)$. Assuming a uniformly distributed quadrature pair phase angle, the maximum approximation error is 0.028A, the mean error is 0.000066A, and the standard deviation about A is 0.00828A. This algorithm is far more accurate than modified versions of Robertson's approximation ($\hat{A} = X + bY$, $b = 1/2$ or $3/8$ or $1/4$) currently being used for most digital signal processing applications. An immediate application is the wideband digital spectrum analyzer under development for monitoring radio frequency interference (RFI) at DSN stations. The algorithm could also be used in digital radar processors.

Digital signal processors sometimes require the computation of the amplitude $A = \sqrt{I^2 + Q^2}$ of an inphase/quadrature component pair, I and Q (e.g., Fourier analyzers, radar receivers). The squaring and square root operations required to determine A exactly are computationally complex, and the intermediate terms, I^2 and Q^2, require double word storage. To simplify this calculation, Robertson (Ref. 1) in 1971 proposed the linear approximation $\hat{A} = X + 1/2\ Y$, where $X \equiv \max(|I|, |Q|)$, $Y \equiv \min(|I|, |Q|)$. The appeal of his scheme is that it can be trivially mechanized by digital shift, compare, and sum operations. Since then, many digital devices have been built incorporating either Robertson's formula or modified versions in which the digital coefficient 1/2 is changed to 3/8 or 1/4 to improve performance (Ref. 2). All of these first-generation algorithms are moderately accurate: assuming a uniformly distributed quadrature pair phase angle, they have maximum errors of 7 to 12 percent, mean errors of 0.6 to 9 percent, and standard deviations about A of 4 to 9 percent (see Table 1).

More recently, several piecewise linear amplitude approximations have been discovered which are far more accurate than those previously mentioned, yet retain almost the same ease of simple digital implementation. In 1974, Braun and Blaser (Ref. 3) published several very accurate piecewise linear amplitude approximations in a British biweekly journal. They introduced a nondigital scale factor c, and approximated cA by the piecewise linear expression

$$c\widehat{A} = \begin{cases} a_1 X + b_1 Y; & X \geqslant kY \\ a_2 X + b_2 Y; & X \leqslant kY \end{cases} \tag{1}$$

where the coefficients of X and Y were restricted to be digital fractions up to 8 bits in length, and k was arbitrarily set to 2. Converting X and Y to polar representation, the fractional amplitude approximation error has the form

$$\epsilon(\theta) \equiv \frac{c\widehat{A} - cA}{cA} = \frac{1}{c}\,(a_i \cos\theta + b_i \sin\theta) - 1 \tag{2}$$

where $\theta \equiv \tan^{-1}(Y/X)\,\epsilon(0, \pi/4)$. The digital coefficients and the scale factor c in Eq. (1) were selected to minimize $\max|\epsilon(\theta)|$. Their results are shown in Table 1 along with mean and rms errors calculated under the earlier assumption that θ is a uniformly distributed random variable.

In 1976, Filip (Ref. 4) published 13 linear approximations for computing A: of these, only two were piecewise linear with digital coefficients. As shown in Table 1, they are not quite as accurate as the best of the Braun/Blaser approximations.

During the past year, while seeking to simplify the data processing hardware of the wideband digital spectrum analyzer being developed for monitoring RFI at DSN stations (Ref. 5), the authors discovered yet another useful approximation. Unaware at the time of the path that had been carved before us by Braun and Blaser and Filip, we independently adopted the piecewise linear approach of Eq. (1) to the problem, restricting $c = 1$ but not preselecting k, and considering digital coefficients up to 3 bits in length. As shown in Table 1, our resulting amplitude approximation is comparable in accuracy to the three best Braun/Blaser formulas. The superiority of these second-generation digital algorithms over their predecessors is illustrated by the plots of $\epsilon(\theta)$ in Fig. 1.

Let us elaborate briefly on the RFI application of this algorithm. As described in Ref. 5, the power in the ith spectral line, based on the jth observation of the input signal, has the form

$$P_{ij} = I_{ij}^2 + Q_{ij}^2 \tag{3}$$

Since the spectrum analyzer is of the multilook category, it makes L independent determinations of each spectral line power. The optimum test for detecting whether the ith spectral line consists of an external signal imbedded in internal noise, or noise alone, is to compare

$$P_i \equiv \frac{1}{L} \sum_{j=1}^{L} P_{ij} \tag{4}$$

with a threshold η, selected to achieve desired false alarm and miss probabilities. However, if the I_{ij}'s and Q_{ij}'s are K-bit words, the P_{ij}'s must be stored as 2K-bit words.

Suppose we elect to use a suboptimal detection scheme in which we average the L amplitudes

$$A_{ij} \equiv \sqrt{I_{ij}^2 + Q_{ij}^2}\,; \quad j = 1, \ldots, L \tag{5}$$

instead of the P_{ij}'s, and compare the random variable

$$A_i \equiv \frac{1}{L} \sum_{j=1}^{L} A_{ij}$$

with a new threshold η'. For sufficiently large L, this suboptimal approach costs us a performance loss of 0.19 dB (Ref. 6). If we use our new algorithm to approximate the A_{ij}'s prior to the averaging operation, the accuracy of the approximation is such that the additional performance loss is negligible. By using the amplitude approximation technique, we avoid the double-word processing required for optimal detection. The resulting hardware simplification represents a significant reduction in processing time and component costs.

References

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Table 1. Performance comparison of digital algorithms for fast amplitude approximation of quadrature pairs (for notation, refer to Eqs. 1 and 2)

Originators	a_1	a_2	b_1	b_2	k	c	max $\lvert\epsilon\rvert$	$\overline{\epsilon}$	$\sqrt{\overline{\epsilon^2}}$
Robertson	1		$\frac{1}{2}$		–	1	0.1180	0.0868	0.0921
	1		$\frac{3}{8}$		–	1	0.0680	0.0402	0.0476
	1		$\frac{1}{4}$		–	1	0.1161	-0.0065	0.0416
Braun/Blaser	$\frac{1}{2}$		$\frac{1}{4}$		–	0.52951	0.0557	0.0262	0.0392
	$\frac{3}{4}$	$\frac{1}{2}$	0	$\frac{1}{2}$	2	0.71041	0.0557	0.0019	0.0338
	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{3}{8}$	2	0.62615	0.0179	0.0023	0.0108
	$\frac{5}{8}$	$\frac{65}{128}$	$\frac{9}{64}$	$\frac{3}{8}$	2	0.63127	0.0148	0.0016	0.0086
	$\frac{19}{32}$	$\frac{1}{2}$	$\frac{9}{64}$	$\frac{11}{32}$	2	0.60196	0.0136	0.0041	0.0080
Filip	1	$\frac{7}{8}$	0	$\frac{1}{2}$	4	1	0.0298	0.0062	0.0123
	1	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	2	1	0.0606	-0.0301	0.0341
Levitt/Morris	1	$\frac{7}{8}$	$\frac{1}{8}$	$\frac{1}{2}$	3	1	0.0277	0.0001	0.0082

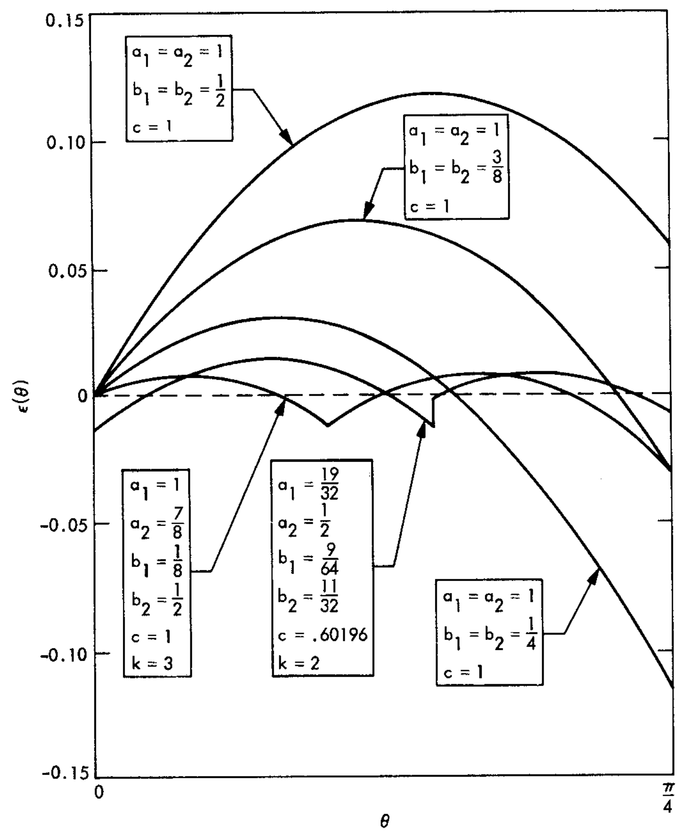

Fig. 1. Comparison of amplitude approximation errors of first- and second-generation digital algorithms (see Eq. 1 for definition of parameters)